# Selected Statistical Properties in the Inner Region of a Transpired Turbulent Boundary Layer

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HISTORICALLY, the study of turbulence phenomena has developed along two quite independent paths. The study of boundary-layer phenomena proceeded from a need to understand surface shear stress and hence the basic analysis was cast in the form that had been successful in laminar flow, resulting in the introduction of an empirical eddy viscosity. Simultaneously, the study of uniform, isotropic turbulence proceeded from a statistical basis yielding significant results concerning the spectral transfer of energy across the wave numbers. It has recently become apparent that solutions are imperative for problems in which the fluctuating characteristics of turbulence in addition to the mean profiles of the variables of interest are of paramount importance in an evaluation of the problem.

It has, thus, become necessary for progress in the engineering solution of many problems in turbulent flow to involve more of the statistical description of the turbulence in the analysis of the problem. New methods for computing turbulent mixing and shear flows are being developed which attempt to overcome the limitations of the phenomenological approaches and which also attempt to incorporate more of the physics of turbulence into the solution. These new techniques have been largely based upon incorporating in one fashion or another the turbulent kinetic energy equation into the set of equations requiring solution. Usually this is accomplished by modeling the various terms in the turbulent kinetic energy equation in terms of the basic properties of turbulence and local flow conditions.

This Note presents the results of a study directed toward the experimental evaluation of the turbulent modeling parameters utilized in the turbulent kinetic energy equation and their application in correlating the mean velocity distribution across the inner region of a turbulent boundary layer with surface mass addition.

The case considered is that of a uniform two-dimensional incompressible turbulent boundary layer developed initially over an impermeable surface with mass injected uniformly over the surface into the boundary layer from a downstream station. The axial pressure gradient is considered zero. For this case, the turbulent kinetic energy equation may be written as

$$\langle U\rangle\,\partial(\langle q^2\rangle/2)/\partial x+\langle V\rangle\,\partial(\langle q^2\rangle/2)/\partial y+ (\langle\rho uv\rangle/\langle\rho\rangle)\,\partial\langle U\rangle/\partial y+ \partial(\langle pv\rangle/\langle\rho\rangle+\langle q^2v\rangle/2)/\partial y+\varepsilon=0 \qquad (1)$$

where $\langle U\rangle$ and $\langle V\rangle$ are the mean velocities in the $x$- and $y$-directions, respectively, $\langle q^2\rangle/2=(\langle u^2\rangle+\langle v^2\rangle+\langle w^2\rangle)/2$ is the turbulent kinetic energy, $p$ is the pressure fluctuation, $\langle\rho\rangle$ is the mean density, $u$, $v$, and $w$ are the velocity fluctuations, and $\varepsilon$ is the dissipation of the turbulent kinetic energy due to viscous effects.

Equation (1) represents the rate of change of the turbulent kinetic energy as the net sum of advection, production, diffusion, and viscous dissipation of the turbulent kinetic energy.

Following Townsend[1] and Bradshaw,[2] the incompressible turbulent kinetic energy equation may be converted into a shear stress equation by defining

$$\tau/\langle\rho\rangle=-\langle\rho uv\rangle/\langle\rho\rangle \qquad (2)$$

$$a_1=\tau/\langle\rho\rangle\langle q^2\rangle \qquad (3)$$

and

$$L=(\tau/\langle\rho\rangle)^{3/2}/\varepsilon \qquad (4)$$

$$(\langle pv\rangle/\langle\rho\rangle+\langle q^2v\rangle/2)=-a_2(\langle q^2\rangle)^{3/2}\,\mathrm{sgn}\,(\partial\langle q^2\rangle/\partial y) \qquad (5)$$

In these expressions, $\tau$ is the local shear stress, $L$ is an energy dissipation length, $a_1$ and $a_2$ are properties of the turbulence field which require experimental evaluation, and the sign of $\partial\langle q^2\rangle/\partial y$ has been included to ensure that the net energy flux is down the gradient of turbulent intensity.

Preliminary evaluation of the statistical properties of an isothermal turbulent boundary layer with surface mass injection have been reported by Peterson and Isaacson.[3] Hot-film constant temperature anemometry equipment was used in the same low speed wind tunnel as described in Ref. 4 to obtain distributions of $\langle -uv\rangle$, $\langle u^2\rangle$, $\langle v^2\rangle$ double-velocity correlations and $\langle u^2v\rangle$ triple-velocity correlations as functions of distance across the boundary layer with surface mass injection as the controlled parameter. The wind tunnel used in this study has a test section 7 in. × 7 in. in cross section and 8 ft in length. The last two feet of the upper wall of the test section are fitted with an inconel porous plate through which air was injected. The free-stream velocity was 25 fps with the pressure gradient through the last four feet of the test section set at approximately zero through the use of an adjustable bottom wall of the wind tunnel.

In the study reported in this Note, profiles of turbulent shear stress, $\langle -uv\rangle$, turbulent kinetic energy, $\langle q^2\rangle/2=(\langle u^2\rangle+2\langle v^2\rangle)/2$, and the turbulent transport of turbulent kinetic energy, $\langle q^2v\rangle/2=(\langle u^2v\rangle+2\langle v^3\rangle)/2$; were obtained at three axial stations through the wind tunnel test section for various values of surface mass addition. The Reynolds numbers for these stations based upon the distance from the entrance to the test section were $Re_{x_1}=8.95\times10^5$, $Re_{x_2}=8.37\times10^5$, and $Re_{x_3}=7.8\times10^5$. The location of the virtual origin of the boundary layer was approximated by extrapolating the displacement thickness variations presented by Al Saji[4] to zero. The Reynolds numbers based upon the distances to the virtual origin for each of the stations where $Re_{xvo_1}=5.94\times10^5$, $Re_{xvo_2}=5.34\times10^5$, and $Re_{xvo_3}=4.78\times10^5$, respectively.

The surface shear stress coefficient for nonblowing conditions $C_{f_o}$ was determined through the use of the following relationship presented by Schlichting.[5]

$$C_{f_o}=(2\log Re_{xvo}-0.65)^{-2.3} \qquad (6)$$

where $Re_{xvo}$ is the Reynolds number based upon the distance from the virtual origin to the particular station. Utilizing the values of $C_{f_o}$ from Eq. (6), the corresponding values of the surface shear stress coefficient with blowing $C_f$ were obtained from Fig. 6 of Simpson, Moffat, and Kays,[6] for the particular

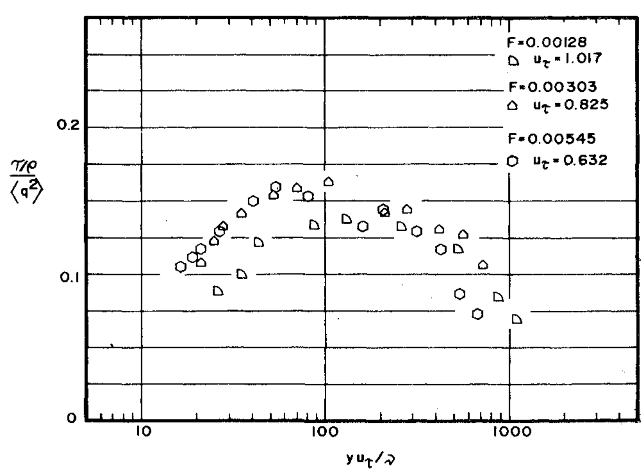


**Fig. 1 Experimental values of the turbulent shear stress modeling parameter, $a_1$; $F=\rho_w v_w/\rho_e u_e$ and $\langle q^2\rangle=\langle u^2\rangle+2\langle v^2\rangle$.**

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values of the blowing parameter $2F/C_{fo}$, where $F = \rho_w v_w/\rho_e u_e$, in which the subscript $e$ indicates the properties evaluated at the outer edge of the boundary layer. These values of $C_f$ were then used to compute the friction velocity defined $u_\tau^2 = u_e^2 C_f/2$.

Figure 1 presents results for the ratio of turbulent shear stress to twice the turbulent kinetic energy [the term $a_1$ defined in Eq. (3)] for various mass injection ratios as a function of $\log yu_\tau/v$, where $v$ is the kinematic viscosity. As can be seen from this figure, there does not appear to be any systematic variation of $a_1$ with blowing rate. These data also are grouped around the value of 0.15 over a significant portion of the boundary layer. We, therefore, set $a_1 = 0.15$ for all of our subsequent correlation calculations.

Figure 2 presents experimental values of the ratio $\langle q^2 v\rangle/(\langle q^2\rangle)^{3/2}$ as a function of $y^+ = yu_\tau/v$ for no-blowing and three different blowing rates. It is apparent from these results, that the ratio $\langle q^2 v\rangle/(\langle q^2\rangle)^{3/2}$ is almost independent of $y^+$, through the inner region up to a value of $y^+ \leq 100$.

Thus, it is consistent with these experimental results to assume that both the parameters $a_1$ and $a_2$ are independent of $y^+$ through the inner region, keeping the provision, however, that $a_2$ is to be taken as dependent upon the blowing rate.

Introducing these parameters into Eq. (1) and neglecting the variation of $\langle u\rangle$ with $x$ so that $\langle V\rangle \cong \langle V_w\rangle$, we may write the turbulent kinetic energy equation as

$$[1/(\tau/\rho)^{1/2}]\,dU - [V_w/2a_1(\tau/\rho)^{3/2}]d(\tau/\rho) \times [3a_2/2(a_1)^{3/2}(\tau/\rho)]d(\tau/\rho) = (1/L)\,dy \quad (7)$$

where all the brackets denoting average quantities have been dropped, and the variation of $\langle q^2\rangle/2$ in the $x$-direction has been neglected.

The mean momentum equation may be used to relate the mean velocity and the local shear stress. For zero axial pressure gradient and a negligible molecular contribution to the shear stress, this may be written as

$$\tau/\rho = u_\tau^2(1+v_w^+ u^+) \quad (8)$$

where

$$u_\tau^2 = \tau_w/\rho \quad (9)$$

$$v_w^+ = V_w/u_\tau \quad (10)$$

and

$$u^+ = U/u_\tau \quad (11)$$

Substituting these expressions into Eq. (7) and integrating the result from the outer edge of the viscous sublayer to a given point in the inner region yields the results

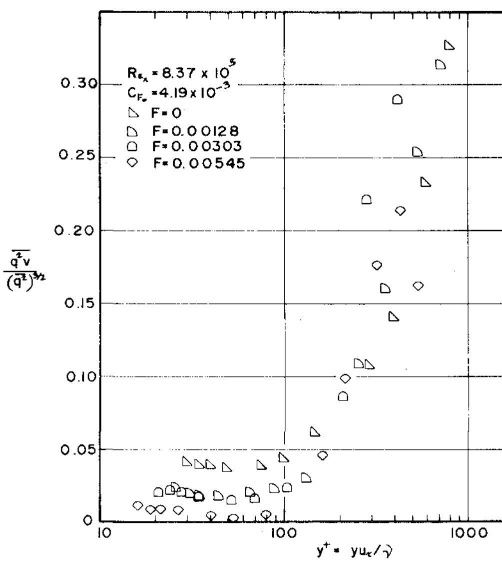


**Fig. 2 Experimental values of the turbulent diffusion transport modeling parameters, $a_2$; $F = \rho_w v_w/\rho_e u_e$ and $\langle q^2 v\rangle = \langle u^2 v\rangle + 2\langle v^3\rangle$.**

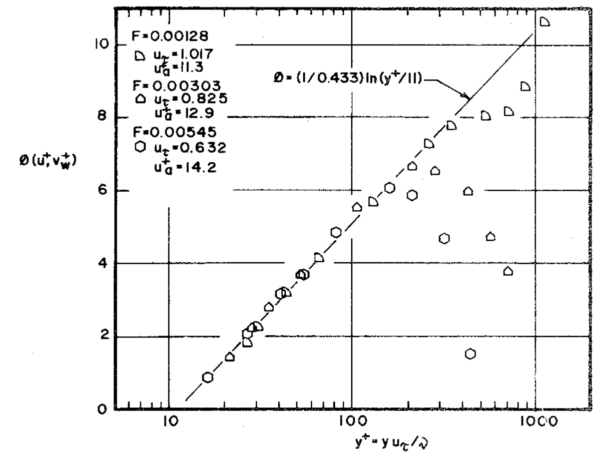


**Fig. 3 Correlation results obtained from integrated turbulent kinetic energy through the inner region. $F = \rho_w v_w/\rho_e u_e$ and $y_a^+ = 11$. The value of $u_a^+$ empirically chosen for the indicated agreement is also presented.**

$$(2/v_w^+)[(1+v_w^+u^+)^{1/2} - (1+v_w^+u_a^+)^{1/2}] + (v_w^+/a_1)[(1+v_w^+u^+)^{-1/2} - (1+v_w^+u_a^+)^{-1/2}] - (3a_2/2(a_1)^{3/2})\ln[(1+v_w^+u^+)/(1+v_w^+u_a^+)] = (1/K)\ln(y^+/y_a^+) \quad (12)$$

where $y_a^+$ is the nondimensionalized distance to the outer edge of the sublayer, $u_a^+$ is the nondimensionalized velocity at the outer edge of the sublayer, and $L^+$ has been set equal to $Ky^+$, where $y^+ = yu_\tau/v$. Equation (12) reduces to the results obtained by Simpson[7] when the terms involving the statistical parameters $a_1$ and $a_2$ are omitted.

Simpson[7] presents results which indicate that $u_a^+$ and $y_a^+$ are essentially independent of blowing and may be obtained from results from unblown flat plate experiments. However, we have found that in order to obtain a correlation of data based on Eq. (12), it is necessary to incorporate empirically determined values for $u_a^+$. We have also found that a correlation of results can be obtained with $y_a^+ = 11$, $a_1 = 0.15$, and with $a_2$ allowed to take on the value measured during the experiment. The resulting equation may be written as

$$\phi = (2/v_w^+)[(1+v_w^+u^+)^{1/2} - (1+v_w^+u_a^+)^{1/2}] + (6.67v_w^+)[(1+v_w^+u^+)^{-1/2} - (1+v_w^+u_a^+)^{-1/2}] - (25.86a_2)\ln[(1+v_w^+u^+)/(1+v_w^+u_a^+)] = (1/K)\ln(y^+/11) \quad (13)$$

This equation has been written with $a_2$ and $u_a^+$ as parameters so that the dependence of these two terms on $v_w^+$ may be included.

Figure 3 presents the left-hand side of Eq. (13) plotted vs $\log yu_\tau/v$ for a series of increasing injection rates. These results indicate a slope of $K = 0.433$, independent of blowing rate. It would appear that the dependency of $K$ on blowing rate reported in Ref. 8 was a result of holding the parameter $a_2$ constant without accounting for the dependency of $a_2$ on blowing rate.

It should be noted that the correlation indicated in Fig. 3 was obtained by empirically determining the appropriate values of $u_a^+$. The resulting values of $u_a^+$ could be approximated by a relationship of the form

$$u_a^+ = 11.0 + 12.4v_w^+ \quad (14)$$

within $\pm 6\%$ of the empirically determined values.

Our objective in this Note has been to present a correlation of experimentally determined mean velocity with distance in a turbulent boundary layer with surface mass injection based upon the turbulent kinetic energy equation with statistical modeling parameters experimentally determined. Once such a correlation has been established, then it may be used to obtain the mean velocity profile for a given turbulent boundary layer with mass injection. The mean momentum equation then yields the distribution of shear stress across the inner region, with the parameter $a_1$ used to calculate the distribution of the turbulent kinetic energy. Finally, the experimentally determined parameter $a_2$, as indicated in Fig. 2, yields the distribution of the turbulent transport of turbulent kinetic energy across the inner region of the boundary layer.

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# Piecewise Uniform Optimum Design for Axial Vibration Requirement


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### Introduction

PIECEWISE uniform design of clamped bars for a fundamental frequency requirement has been studied by Turner,[1] Sheu,[2] Sippel and Warner.[3] However, the methods used by these authors do not permit one to find an exact solution for more than two uniform regions in the rod in any reasonably practical way because of the rapid increase in complexity of the general frequency relation. Matrix and steepest gradient methods have been used to get an approximate solution. Instead, the objective of this Note is to show that a closed form of the exact solution exists for an arbitrary number of regions. Then it is used to show the convergence of discrete optimization to the continuous one.

### 1. Governing Equations

The system (Fig. 1) consists of $n$ segments, length $L_i$, linear mass density $m_i$. For axial vibration of fundamental circular

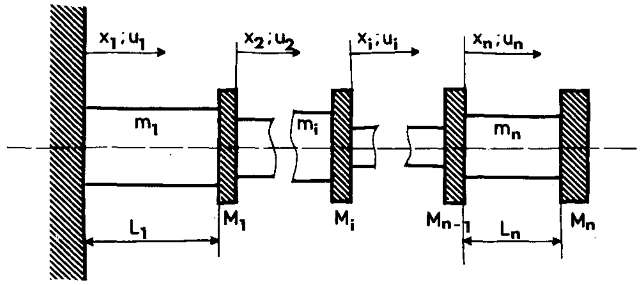


**Fig. 1 Clamped bar with $n$ uniform elements.**

frequency $\omega$, the mode shape for each member is $u_i(x_i)$, $(0 \leqq x_i \leqq L_i)$, given by the modal equation and boundary conditions

$$u_i''(x_i)+\beta^2 u_i(x_i) = 0 \qquad (i = 1,\dots,n)$$

$$u_{i-1}(L_{i-1}) = u_i(0) \qquad (i = 2,\dots,n)$$

$$m_{i-1}u'(L_{i-1})-\beta^2 M_{i-1}u_{i-1}(L_{i-1})-m_i u_i'(0) = 0 \qquad (i = 2,\dots,n)$$

$$u_1(0) = 0$$

$$m_n u_n'(L_n)-\beta^2 M_n u_n(L_n) = 0$$

where $\beta^2 = \omega^2\rho/E$, $\rho$ = material density, $E$ = Young's modulus. Solutions can be expressed as

$$u_i = k_i \sin(\beta x_i+\Phi_i)$$

where

$$0 \leqq x_i \leqq L_i \quad \text{and} \quad 0 \leqq \Phi_i < \pi$$

The $k_i$'s can be eliminated, and the boundary conditions become

$$m_{i+1} = [m_i \cot(\beta L_i+\Phi_i)-\beta M_i]\tan\Phi_{i+1} \qquad (i = 1,\dots,n-1)$$

For the fundamental mode, $u_i > 0$ for $0 < x_i < L_i$ $(i = 1,\dots,n)$. It is easy to prove that this condition implies

$$0 < \beta L_i+\Phi_i < \pi/2 \qquad (i = 1,\dots,n)$$

### 2. Optimization Procedure

We let

$$L = \sum_{i=1}^{n} L_i, \qquad M = \sum_{i=1}^{n} M_i, \qquad \alpha_i = \beta L_i$$

and we nondimensionalize in the following way:

$$l_i = L_i/L, \qquad \Omega_i = \beta M_i L/M, \qquad \mu_i = m_i L/M$$

The optimization problem, which is to minimize the total bar structural mass for the given fundamental frequency $\omega$, can be formulated as follows. Minimize

$$\mathscr{M} = \sum_{i=1}^{n} l_i \mu_i$$

in the $2n$-dimensional $(\mu_i, \Phi_i)$ space, subject to the constraints

$$\mu_i\cot\Phi_i-\mu_{i-1}\cot(\alpha_{i-1}+\Phi_{i-1})+\Omega_{i-1} = 0 \qquad (i = 2,\dots,n) \quad (1)$$

$$\Phi_1 = 0 \quad (2)$$

$$-\mu_n\cot(\alpha_n+\Phi_n)+\Omega_n = 0 \quad (3)$$

$$\Phi_i \geqq 0 \quad (4)$$

$$\alpha_i+\Phi_i < \pi/2 \quad (5)$$

$$\mu_i > 0 \quad (6)$$

This problem is equivalent to finding the unconstrained minimum of the new function

$$W = \sum_{i=1}^{n} l_i\mu_i+\lambda_1(\mu_2\cot\Phi_2-\mu_1\cot\Phi_1+\Omega_1)+ \sum_{i=3}^{n}\lambda_{i-1}[\mu_i\cot\Phi_i-\mu_{i-1}\cot(\alpha_{i-1}+\Phi_{i-1})+\Omega_{i-1}]+ \lambda_n[\Omega_n-\mu_n\cot(\alpha_n+\Phi_n)]$$


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